Lorentz covariant reduced-density-operator theory for relativistic quantum information processing

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Abstract

In this paper, we derived Lorentz covariant quantum Liouville equation for the density operator which describes the relativistic quantum information processing from Tomonaga-Schwinger equation and an exact formal solution for the reduced-density-operator is obtained using the projector operator technique and the functional calculus. When all the members of the family of the hypersurfaces become flat hyperplanes, it is shown that our results agree with those of non-relativistic case which is valid only in some specified reference frame. The formulation presented in this work is general and might be applied to related fields such as quantum electrodynamics and relativistic statistical mechanics.

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Recently, there has been growing interest in the relativistic formulation [1]- [7] of quantum operations for possible near future applications to relativistic quantum information processing such as teleportation [8], entanglement-enhanced communication [9], and quantum clock synchronization [10], [11].

In the non-relativistic case, the key element for studying quantum information processing is the density operator of a quantum register which is derived from the solution of a quantum Liouville equation (QLE) [12], [13] for the total system including an environment. The QLE is an integro-differential equation and it is in general nontrivial to obtain the solution of the form

$$\rho \xrightarrow{\mathcal{E}} \rho' = \hat{\mathcal{E}}[\rho], \tag{1}$$

where ρ is the reduced density operator of the quantum register and $\hat{\mathcal{E}}$ is the superoperator describing the evolution of ρ by the quantum information processing. In the previous works, we have employed a time-convolutionless reduced-density-operator formalism to model quantum devices [14] and noisy quantum channels [15], [16].

The first step toward the relativistic quantum information theory would be the formulation of Lorentz covariant QLE and the derivation of the reduced-density-operator which is a solution of the covariant QEL. The goal of this paper is to derive Lorentz covariant quantum Liouville equation which describes the relativistic quantum information processing and obtain a formal solution for the reduced-density-operator pertaining to the system (or electrons) part alone.

It is well known that neither the non-relativistic Schrödinger equation nor the QLE is Lorentz covariant. As a result, it is expected that the usual non-relativistic definition of the reduced-density-operator and its functionals such as quantum entropy have no invariant meaning in special relativity. Another conceptual barrier for the relativistic treatment of quantum information processing is the difference of the role played by the wave fields and the state vectors in the quantum field theory. In non-relativistic quantum mechanics both the wave function and the state vector in Hilbert space give the probability amplitude which can be used to define conserved positive probability densities or density matrix. On the other hands, in relativistic quantum field theory, covariant wave fields are not probability amplitude at all, but operators which create or destroy particles in spanned by states defined as containing definite numbers of particles or antiparticles in each normal mode [17]. The role of the fields is to make the interaction or S-matrix satisfy the Lorentz invariance and the cluster decomposition principle. The information of the particle states is contained in the state vectors of the Hilbert space spanned by states containing $0, 1, 2, \cdots$ particles as in the case of non-relativistic quantum mechanics. So it seems like that one needs to obtain the covariant equation of motion for the state vector and derive the covariant QLE out of it.

Some fifty years ago, Tomonaga [18] and Schwinger [19] derived a covariant equation of motion for the quantum state vector in terms of the functional derivative, known as Tomonaga-Schwinger (T-S) equation,

$$i\frac{\delta\Psi[\sigma]}{\delta\sigma(x)} = \mathcal{H}_{int}(x)\Psi[\sigma], \tag{2}$$

in the interaction picture. Here x is a space-time four-vector, σ is the spacelike hypersurface, $\Psi[\sigma]$ is the state vector which is a functional of σ, $\mathcal{H}_{int}(x) = \mathcal{H}_{int}[\varphi_\alpha(x)]$ is the interaction Hamiltonian density which is a functional of quantum field $\varphi_\alpha[x]$, and $\frac{\delta}{\delta\sigma(x)}$ is the Lorentz invariant functional derivative [20]. The functional derivative of $\Psi[\sigma]$ is defined as

$$\frac{\delta\Psi[\sigma]}{\delta\sigma(x)} = \lim_{\delta\omega\to 0}\frac{\Psi[\sigma'] - \Psi[\sigma]}{\delta\omega}, \tag{3}$$

where $\delta\omega$ is an infinitesimal four-dimensional volume between two hypersurfaces σ and σ'. The formal solution of equation (3) is given by

$$\Psi[\sigma] = \mathcal{U}[\sigma, \sigma_0]\Psi[\sigma_0], \tag{4}$$

where the generalized transformational functional satisfies the T-S equation

$$i\frac{\delta\mathcal{U}[\sigma, \sigma_0]}{\delta\sigma(x)} = \mathcal{H}_{int}(x)\mathcal{U}[\sigma, \sigma_0] \tag{5}$$

with the boundary condition $\mathcal{U}[\sigma_0, \sigma_0] = 1$. The generalized transformation functional $\mathcal{U}[\sigma, \sigma_0]$ is a unitary operator. We also have [19]

$$\frac{\delta \mathcal{U}^{-1}[\sigma, \sigma_0]}{\delta \sigma(x)} = -\mathcal{U}^{-1}[\sigma, \sigma_0] \frac{\delta \mathcal{U}[\sigma, \sigma_0]}{\delta \sigma(x)} \mathcal{U}^{-1}[\sigma, \sigma_0], \tag{6}$$

from the unitary condition. Throughout the paper, we assume $\hbar = c = 1$. The expectation value of some field variable $F(x)$ becomes

$$\begin{aligned} < F(x) > &= (\Psi[\sigma], F(x)\Psi[\sigma]) \\ &= trace(F(x)\Psi[\sigma]\Psi^\dagger[\sigma]) \\ &= trace(F(x)\rho_T[\sigma]). \end{aligned} \tag{7}$$

¿From equation (7), we notice that the total density operator $\rho_T[\sigma]$ can be written as

$$\begin{aligned} \rho_T[\sigma] &= \Psi[\sigma]\Psi^\dagger[\sigma] \\ &= \mathcal{U}[\sigma, \sigma_0]\Psi[\sigma]\Psi^\dagger[\sigma]\mathcal{U}^{-1}[\sigma, \sigma_0]. \end{aligned} \tag{8}$$

Then,

$$\begin{aligned} \frac{\delta \rho_T[\sigma]}{\delta \sigma(x)} &= \frac{\delta}{\delta \sigma} \{ \mathcal{U}[\sigma, \sigma_0]\Psi[\sigma_0]\Psi^\dagger[\sigma_0]\mathcal{U}^{-1}[\sigma, \sigma_0] \} \\ &= [\frac{\delta \mathcal{U}[\sigma, \sigma_0]}{\delta \sigma(x)} \mathcal{U}^{-1}[\sigma, \sigma_0], \rho_T[\sigma]] \\ &= -i[\mathcal{H}_{int}(x), \rho_T[\sigma]] \\ &= -i\hat{\mathcal{L}}(x)\rho_T[\sigma], \end{aligned} \tag{9}$$

where $\hat{\mathcal{L}}(x)$ is the Liouville superoperator. Since equation (9) describes the Lorentz covariant equation of motion for the total density operator, we denote it as the covariant quantum Liouville equation (CQLE). Note that the Liouville superoperator is not an operator in the Hilbert space of state vectors but a linear operator in the Hilbert-Schmidt space of density matrices [16]. Here $\rho_T[\sigma]$ contains the information for the total system, for example, an interacting spin-$\frac{1}{2}$ massive particles and photons in the case of quantum electrodynamics (QED).

In order to extract the information of the system or the electrons alone, it is convenient to use the projection operators [12], [22], [23] that decompose the total system by eliminating the degrees of freedom for the environment, say, the photon field in the case of QED. The information of the system is then contained in the reduced-density-operator $\rho[\sigma]$ which is defined as

$$\begin{aligned}\rho[\sigma] &= tr_B \rho_T[\sigma] \\ &= tr_B \mathcal{P} \rho_T[\sigma],\end{aligned} \tag{10}$$

where the projection operator $\mathcal{P}$ and $\mathcal{Q}$ are defined as $\mathcal{P}X = \rho_B tr_B(X)$, $\mathcal{Q} = 1 - \mathcal{P}$, for any covariant dynamical variable X, ρ_B is the density matrix for the quantum environment at σ_0 and tr_B indicates a partial trace over the quantum environment. The projection operators satisfy the operator identities $\mathcal{P}^2 = \mathcal{P}$, $\mathcal{Q}^2 = \mathcal{Q}$, $\mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P} = 0$ and $[\frac{\delta}{\delta\sigma(x)}, \mathcal{P}] = [\frac{\delta}{\delta\sigma(x)}, \mathcal{Q}] = 0$. Furthermore, we would like to note that $(\frac{\delta}{\delta\sigma(x)})^{-1} = \int d^4x$ [20], and the system and the environment are decoupled at σ_0. We also note that the projection operators $\mathcal{P}$ and $\mathcal{Q}$ are functionals of the initial hypersurface $\sigma_0(\neq \sigma$ for all $x)$ and unless otherwise specified, we will omit the functional argument. However, one needs to keep track of the functional argument especially in the four-dimensional integration.

The CQLE (9) can be decomposed into two coupled equation for $\mathcal{P}\rho_T[\sigma]$ and $\mathcal{Q}\rho_T[\sigma]$:

$$\frac{\delta}{\delta\sigma(x)} \mathcal{P}\rho_T[\sigma] = -i\mathcal{P}\hat{\mathcal{L}}(x)\mathcal{P}\rho_T[\sigma] - i\mathcal{P}\hat{\mathcal{L}}(x)\mathcal{Q}\rho_T[\sigma], \tag{11a}$$

$$\frac{\delta}{\delta\sigma(x)} \mathcal{Q}\rho_T[\sigma] = -i\mathcal{Q}\hat{\mathcal{L}}(x)\mathcal{Q}\rho_T[\sigma] - i\mathcal{Q}\hat{\mathcal{L}}(x)\mathcal{P}\rho_T[\sigma], \tag{11b}$$

The formal solution of (11b) is given by [24]

$$\mathcal{Q}\rho_T[\sigma] = \{\theta[\sigma] - 1\}\mathcal{P}\rho_T[\sigma], \tag{12}$$

where

$$\theta^{-1}[\sigma] = 1 + i\int_{\sigma_0}^{\sigma} d^4x' H[\sigma(x), \sigma(x')]\mathcal{Q}\hat{\mathcal{L}}(x')\mathcal{P}G_R[\sigma(x), \sigma(x')], \tag{13}$$

$$H[\sigma(x), \sigma(x')] = T \exp\{-i \int_{\sigma(x')}^{\sigma(x)} d^4x'' \mathcal{Q}\hat{\mathcal{L}}(x'')\mathcal{Q}\}, \tag{14}$$

and

$$G_R[\sigma(x), \sigma(x')] = T^c \exp\{i \int_{\sigma(x')}^{\sigma(x)} d^4x'' \hat{\mathcal{L}}(x'')\}. \tag{15}$$

Here T and T^c are time-ordering and anti time-ordering operators, respectively, $H[\sigma, \sigma_0]$ is the projected propagator, and $G_R[\sigma, \sigma_0]$ is the retarded propagator. Once the solution for $\mathcal{Q}\rho_T[\sigma]$ is obtained, it is substituted for the equation for $\mathcal{P}\rho_T[\sigma]$. Then, after, some mathematical manipulations, we obtain

$$\mathcal{P}\rho_T[\sigma] = W^{-1}[\sigma, \sigma_0]\hat{\mathcal{U}}_s[\sigma, \sigma_0]\mathcal{P}\rho_T[\sigma_0], \tag{16}$$

or

$$\rho[\sigma] = tr_B\{W^{-1}[\sigma, \sigma_0]\hat{\mathcal{U}}_s[\sigma, \sigma_0]\rho_B\}\rho[\sigma_0], \tag{17}$$

where

$$W[\sigma, \sigma_0] = 1 + i \int_{\sigma_0}^{\sigma} d^4x' \hat{\mathcal{U}}_s[\sigma(x), \sigma(x')]\mathcal{P}\hat{\mathcal{L}}(x')\{\theta[\sigma(x')] - 1\}\mathcal{P}G_R[\sigma(x), \sigma(x')]\theta[\sigma(x)], \tag{18}$$

and

$$\hat{\mathcal{U}}_s[\sigma, \sigma_0] = T \exp\{-i \int_{\sigma_0}^{\sigma} d^4x' \mathcal{P}\hat{\mathcal{L}}(x')\mathcal{P}\}. \tag{19}$$

Here $\hat{\mathcal{U}}_s[\sigma, \sigma_0]$ is the generalized transformation functional or the propagator for the reduced system. Mathematical details of the derivation of equation (12) to (19) involving functional calculus [20] will be presented elsewhere.

It is remarkable to note when hypersurfaces σ_0 and all the members of the family $\{\sigma\}$ are hyperplane flat surfaces parametrized by $t = constant$ [20], then the transformation functional such as $\mathcal{U}_s[\sigma(x), \sigma(x')]$ can be written as $\mathcal{U}_s(t, t')$. As a result, the covariant forms of equations (16) to (19) become reduced to those of the non-relativistic case which is valid only in some specified reference frame given by equations (18) to (25) of reference [15].

By comparing, equations (1) and (17), the covariant superoperator for the relativistic quantum operation $\hat{\mathcal{E}}[\sigma, \sigma_0]$ can be written as

$$\hat{\mathcal{E}}[\sigma, \sigma_0] = tr_B\{W^{-1}[\sigma, \sigma_0]\hat{\mathcal{U}}_s[\sigma, \sigma_0]\rho_B\}. \tag{20}$$

So far all our results are exact and the equations (16) to (20) would be the key steps in the analysis of relativistic quantum information processing. Apart from describing quantum information processing, QLE and reduced-density-operator have been essential in solving various quantum optics and non-Markovian optical problems in the non-relativistic domain [12]- [14]. So it might be interesting to extend this approach to revisit relativistic quantum electrodynamics problems, which were solved relying on renormalization procedures in field theory, using the covariant form of quantum Liouville equation. On the other hand, relativistic thermodynamics or statistical mechanics look like an area where the knowledge of the density operator or the reduced-density-operator might come in handy provided the ambiguity of the temperature concept in special relativity is resolved. We believe our formalism is general and could be applied to related fields such as QED and relativistic statistical mechanics.

In summary, we have derived Lorentz covariant quantum Liouville equations for the density operator in functional of hypersurface from T-S equation and obtained formal solution for the reduced-density-operator which is also in covariant form using the projection operator technique and the functional calculus. When all the members of the family of the hypersurfaces become flat hyperplanes, our results agree with those of the non-relativistic case. Our formulation is exact and general so it could be applied not only to the relativistic quantum information processing but also to the related fields such as QED, field theory and relativistic statistical mechanics.

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